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Global Existence for a Class of System of Nonlinear Wave and Klein-Gordon Equations

By

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Abstract

We consider the Cauchy problem for systems of nonlinear wave and Klein-Gordon equations in three space dimensions. We introduce a new kind of null condition for this system to ensure the global existence of small solutions whose wave components decay faster than those in the previous work.

§ 1. Introduction

Let $\Box = \partial_t^2 - \Delta$. An equation of the form $(\Box + m^2)\phi = \Phi$ is called the wave equation when $m = 0$, and the Klein-Gordon equation when $m > 0$.

We consider the Cauchy problem for systems of nonlinear wave and Klein-Gordon equations in three space dimensions:

$$(\Box + m_j^2)u_j = F_j(u, \partial u) \quad \text{in } (0,\infty)\times\mathbb{R}^3, \tag{1.1}$$

$$u_j(0,x) = \varepsilon f_j(x),\ (\partial_t u_j)(0,x) = \varepsilon g_j(x), \quad x \in \mathbb{R}^3 \tag{1.2}$$

for $j = 1, 2, \ldots, N$, where m_j's are non-negative constants,

$$u = u(t,x) = \big(u_1(t,x), \ldots, u_N(t,x)\big)$$

is an $\mathbb{R}^N$-valued unknown function, and

$$\partial u = (\partial_0 u, \partial_1 u, \partial_2 u, \partial_3 u) := (\partial_t u, \partial_{x_1} u, \partial_{x_2} u, \partial_{x_3} u).$$

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The nonlinear term $F = (F_1, \ldots, F_N)$ is a smooth function of $(u, \partial u)$ with

$$F(u, \partial u) = O(|u|^2 + |\partial u|^2) \tag{1.3}$$

near $(u, \partial u) = (0, 0)$. We suppose that

$$f = (f_1, \ldots, f_N), g = (g_1, \ldots, g_N) \in \mathscr{S}(\mathbb{R}^3; \mathbb{R}^N),$$

where $\mathscr{S}(\mathbb{R}^3; \mathbb{R}^N)$ is the space of $\mathbb{R}^N$-valued rapidly decreasing functions on $\mathbb{R}^3$. ε is a small positive parameter.

We assume that there is $N_0 \in \{0, 1, \ldots, N\}$ such that

$$m_j > 0 \text{ for } 1 \le j \le N_0, \text{ and } m_j = 0 \text{ for } N_0 + 1 \le j \le N. \tag{1.4}$$

This relation will be understood as $m_j > 0$ for all j when $N_0 = N$, and $m_j = 0$ for all j when $N_0 = 0$. We put $v_j = u_j$ for $1 \le j \le N_0$, and $w_j = u_j$ for $N_0 + 1 \le j \le N$. v_j is called the Klein-Gordon component, and w_j is called the wave component. We write $u = (v, w)$ with $v = (v_1, \ldots, v_{N_0})$ and $w = (w_{N_0+1}, \ldots, w_N)$.

Before we state some known results, we introduce some notation. Let $F^{\mathrm{q}} = \left(F_1^{\mathrm{q}}, \ldots, F_N^{\mathrm{q}}\right)$ be the quadratic part of F, that is to say

$$F^{\mathrm{q}}(u, \partial u) = \lim_{\lambda \to 0} \lambda^{-2} F(\lambda u, \lambda \partial u).$$

For $1 \le j \le N$, we write

$$F_j^{\mathrm{q}}(u, \partial u) = \sum_{1 \le k \le l \le N} F_j^{kl}[u_k, u_l] \tag{1.5}$$

with

$$F_j^{kl}[u_k, u_l] = \sum_{|\alpha|, |\beta| \le 1} c_j^{kl\alpha\beta} (\partial^\alpha u_k)(\partial^\beta u_l), \tag{1.6}$$

where $c_j^{kl\alpha\beta}$ are real constants satisfying $c_j^{kk\alpha\beta} = c_j^{kk\beta\alpha}$. For $1 \le j \le N$ we set

$$\begin{aligned}
F_j^{\mathrm{K}}(v, \partial v) &:= \sum_{1 \le k \le l \le N_0} F_j^{kl}[v_k, v_l], \\
F_j^{\mathrm{KW}}(u, \partial u) &:= \sum_{1 \le k \le N_0,\ N_0+1 \le l \le N} F_j^{kl}[v_k, w_l], \\
F_j^{\mathrm{W}}(w, \partial w) &:= \sum_{N_0+1 \le k \le l \le N} F_j^{kl}[w_k, w_l].
\end{aligned}$$

Then F_j^{q} is decomposed into three parts:

$$F_j^{\mathrm{q}}(u, \partial u) = F_j^{\mathrm{K}}(v, \partial v) + F_j^{\mathrm{KW}}(u, \partial u) + F_j^{\mathrm{W}}(w, \partial w).$$

We define $F_j^{\mathrm{h}}(u,\partial u)=F_j(u,\partial u)-F_j^{\mathrm{q}}(u,\partial u)$.

We write $F=(F_{\mathrm{K}},F_{\mathrm{W}})$ with

$$F_{\mathrm{K}}=(F_1,\dots,F_{N_0}),\quad F_{\mathrm{W}}=(F_{N_0+1},\dots,F_N).$$

Similarly, for $*=\mathrm{q},\mathrm{K},\mathrm{KW},\mathrm{W},\mathrm{h}$, we write $F^*=(F_1^*,\dots,F_N^*)=(F_{\mathrm{K}}^*,F_{\mathrm{W}}^*)$ with

$$F_{\mathrm{K}}^*=(F_1^*,\dots,F_{N_0}^*),\quad F_{\mathrm{W}}^*=(F_{N_0+1}^*,\dots,F_N^*).$$

Using these notations, and writing

$$(\Box+\mathcal{M}^2)v:=\big((\Box+m_1^2)v_1,\dots,(\Box+m_{N_0}^2)v_{N_0}\big)$$

and $\Box w=(\Box w_{N_0+1},\dots,\Box w_N)$, we can write (1.1) as

$$\begin{aligned}(\Box+\mathcal{M}^2)v&=F_{\mathrm{K}}^{\mathrm{K}}(v,\partial v)+F_{\mathrm{K}}^{\mathrm{KW}}(u,\partial u)+F_{\mathrm{K}}^{\mathrm{W}}(w,\partial w)+F_{\mathrm{K}}^{\mathrm{h}}(u,\partial u),\\ \Box w&=F_{\mathrm{W}}^{\mathrm{K}}(v,\partial v)+F_{\mathrm{W}}^{\mathrm{KW}}(u,\partial u)+F_{\mathrm{W}}^{\mathrm{W}}(w,\partial w)+F_{\mathrm{W}}^{\mathrm{h}}(u,\partial u).\end{aligned}$$

Now let us mention the known results. To do so, we introduce some conditions.

(N1) The null condition for $F_{\mathrm{W}}^{\mathrm{W}}=F_{\mathrm{W}}^{\mathrm{W}}(w,\partial w)=F_{\mathrm{W}}^{\mathrm{W}}(w,\partial_t w,\partial_1 w,\partial_2 w,\partial_3 w)$: We have

$$F_{\mathrm{W}}^{\mathrm{W}}(X,\omega_0 Y,\omega_1 Y,\omega_2 Y,\omega_3 Y)=0 \tag{1.7}$$

for all $X,Y\in\mathbb{R}^{N-N_0}$ and all $\omega=(\omega_1,\omega_2,\omega_3)\in\mathbb{S}^2$ with $\omega_0=-1$.

(A1) $F^{\mathrm{KW}}(u,\partial u)=\big(F_{\mathrm{K}}^{\mathrm{KW}}(u,\partial u),F_{\mathrm{W}}^{\mathrm{KW}}(u,\partial u)\big)$ is independent of w itself[1]; in short, we have $F^{\mathrm{KW}}=F^{\mathrm{KW}}(v,\partial u)$.

(A2) $F_{\mathrm{K}}^{\mathrm{W}}(w,\partial w)$ is independent of w itself; in short, we have $F_{\mathrm{K}}^{\mathrm{W}}=F_{\mathrm{K}}^{\mathrm{W}}(\partial w)$.

In [5], the author proved that if (N1), (A1), and (A2) are fulfilled, then, for any $f,g\in\mathscr{S}(\mathbb{R}^3;\mathbb{R}^N)$, there is a positive constant ε_0 such that the Cauchy problem (1.1)–(1.2) admits a unique global solution, provided that $0<\varepsilon\le\varepsilon_0$ (see Georgiev [3] for a partial result under the strong null condition; see also LeFloch–Ma [14] for an alternative proof, which requires less regularity of the initial data at the cost of the compactness assumption on their support).

For the case of systems of wave equations, namely for the case of $N_0=0$, we have $u=w$ and $F_{\mathrm{W}}^{\mathrm{W}}=F^{\mathrm{q}}$. If we neglect the meaningless conditions, then the above result in [5] can be read as follows: For systems of nonlinear wave equations with (1.3), if

$$F^{\mathrm{q}}(X,\omega_0 Y,\omega_1 Y,\omega_2 Y,\omega_3 Y)=0 \tag{1.8}$$

[1] If we say that "a function G is independent of w itself", it means that G is independent of $w=(w_k)$, but can depend on the derivatives $\partial w=(\partial_a w_k)$.

holds for all $X, Y \in \mathbb{R}^N$ and all $\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{S}^2$ with $\omega_0 = -1$, then we have the small data global existence for $\mathscr{S}$-data. The assumption here is nothing but the null condition for the wave equations introduced by Klainerman [11], in which the small data global existence was proved under the null condition (see also Christodoulou [2]). For the case of Klein-Gordon equations, namely for the case of $N_0 = N$, we have $u = v$, and if we neglect the meaningless conditions, then the above result in [5] can be read as follows: For systems of nonlinear Klein-Gordon equations with (1.3), we have the small data global existence of solutions without any further assumptions. This coincides with the global existence result proved by Klainerman [10] and Shatah [15].

Quite recently, assuming (N1) and (A1), but without assuming (A2), the author [6] proved the small data global existence for (1.1)–(1.2) for C_0^∞-data. Note that we explicitly use the compactness assumption on the support of the initial data to remove the condition (A2), and it is not known whether we can remove (A2) also for $\mathscr{S}$-data.

As for the decay of the solutions for $\mathscr{S}$-data, it was proved in [5] that for any $\delta > 0$, there are positive constants C and ε_0 such that we have

$$|v(t,x)| + |\partial v(t,x)| \le C\varepsilon \langle t + |x| \rangle^{-3/2}, \tag{1.9}$$

$$|w(t,x)| \le C\varepsilon \langle t + |x| \rangle^{-1+\delta} \langle t - |x| \rangle^{-\delta}, \tag{1.10}$$

$$|\partial w(t,x)| \le C\varepsilon \langle x \rangle^{-1} \langle t - |x| \rangle^{-1} \tag{1.11}$$

under the conditions (N1), (A1), and (A2), provided that $0 < \varepsilon \le \varepsilon_0$, where $\langle z \rangle = \sqrt{1 + |z|^2}$. If the initial data belong to $C_0^\infty(\mathbb{R}^3)$, then (1.10) is improved to

$$|w(t,x)| \le C\varepsilon \langle t + |x| \rangle^{-1} \tag{1.12}$$

under the conditions (N1) and (A1) (see [6]). This improvement is due to the compactness assumption on the support of the initial data.

Among the nonlinear terms, $F_{\mathrm{W}}^{\mathrm{K}}(v, \partial v)$ is the worst one to determine the decay rates in (1.10), (1.11), and (1.12). In this paper, we consider the case of $\mathscr{S}$-data, and we would like to introduce a kind of null condition for $F_{\mathrm{W}}^{\mathrm{K}}(v, \partial v)$ to ensure the better decay rate for w than those in (1.10), (1.11), and (1.12).

To introduce the condition, we define

$$\mathbb{H} := \left\{ \vec{\omega} = (\omega_0, \omega_1, \omega_2, \omega_3) \in \mathbb{R}^4; \omega_0^2 - \omega_1^2 - \omega_2^2 - \omega_3^2 = 1 \right\}.$$

We use the notation

$$\vec{\omega}^\alpha = \omega_0^{\alpha_0} \omega_1^{\alpha_1} \omega_2^{\alpha_2} \omega_3^{\alpha_3}$$

for a multi-index $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$.

Definition 1.1 (The null condition for $F_{\mathrm{W}}^{\mathrm{K}}$). We say that the null condition for

$F_{\mathrm{W}}^{\mathrm{K}}$ is satisfied if we have

$$\sum_{|\alpha|,|\beta|\le 1} c_j^{kl\alpha\beta}(im_k\vec{\omega})^\alpha(\overline{im_l\vec{\omega}})^\beta = 0 \tag{1.13}$$

for any $N_0+1 \le j \le N$, any $1 \le k,l \le N_0$ with $m_k = m_l(>0)$, and any $\vec{\omega} = (\omega_0,\omega_1,\omega_2,\omega_3) \in \mathbb{H}$, where the constants $c_j^{kl\alpha\beta}$ are from (1.6).

The following is the main result.

Theorem 1.2. *Let* (1.3) *and* (1.4) *be fulfilled. We assume the conditions* (N1), (A1), (A2), *and*

(N2) *The null condition for $F_{\mathrm{W}}^{\mathrm{K}}$ is satisfied.*

Let $0<\kappa<1/2$. Then, for any $f,g \in \mathscr{S}(\mathbb{R}^3;\mathbb{R}^N)$, there is a positive constant ε_0 such that the Cauchy problem (1.1)–(1.2) *admits a unique global solution $u \in C^\infty\big([0,\infty)\times\mathbb{R}^3;\mathbb{R}^N\big)$, provided that $0<\varepsilon\le\varepsilon_0$. Moreover, there is a positive constant C such that we have* (1.9) *and*

$$|w(t,x)| \le C\varepsilon \langle t+|x|\rangle^{-1}\langle t-|x|\rangle^{-\kappa}, \tag{1.14}$$

$$|\partial w(t,x)| \le C\varepsilon \langle x\rangle^{-1}\langle t-|x|\rangle^{-1-\kappa} \tag{1.15}$$

for $0<\varepsilon\le\varepsilon_0$.

§ 2. Preliminaries

For $z \in \mathbb{R}^d$, we define $\langle z\rangle = \sqrt{1+|z|^2}$.

We introduce

$$\begin{aligned} L_j &:= x_j\partial_t + t\partial_j, && 1\le j\le 3,\\ \Omega_{jk} &:= x_j\partial_k - x_k\partial_j, && 1\le j,k\le 3, \end{aligned}$$

and we set

$$\Gamma := (\Gamma_1,\dots,\Gamma_{10}) = (L,\Omega,\partial) = \big((L_j)_{1\le j\le 3},(\Omega_{jk})_{1\le j<k\le 3},(\partial_a)_{0\le a\le 3}\big).$$

For a multi-index $\alpha=(\alpha_1,\dots,\alpha_{10})$, we write $\Gamma^\alpha = \Gamma_1^{\alpha_1}\cdots\Gamma_{10}^{\alpha_{10}}$. For a smooth function $\phi=\phi(t,x)$ and a non-negative integer s, we define

$$|\phi(t,x)|_s = \sum_{|\alpha|\le s}|\Gamma^\alpha\phi(t,x)|, \quad \|\phi(t)\|_s = \big\||\phi(t,\cdot)|_s\big\|_{L^2(\mathbb{R}^3)}. \tag{2.1}$$

One can easily check that

$$[\Box+m^2, L_j] = [\Box+m^2,\Omega_{jk}] = [\Box+m^2,\partial_a] = 0$$

for $1 \le j, k \le 3$, $0 \le a \le 3$, and $m \ge 0$, where $[A, B] = AB - BA$ for operators A and B. Hence, for any multi-index α, we get

$$(\Box + m^2)(\Gamma^\alpha \phi) = \Gamma^\alpha \big((\Box + m^2)\phi\big).$$

We also have

$$[\Gamma_j, \Gamma_k] = \sum_{l=1}^{10} c_{jkl}\Gamma_l, \quad [\Gamma_j, \partial_a] = \sum_{b=0}^{3} d_{jab}\partial_b$$

for $1 \le j, k \le 10$ and $0 \le a \le 3$ with appropriate constants c_{jkl} and d_{jab}. Consequently, for any non-negative integer s, there is a positive constant C_s such that we have

$$\frac{1}{C_s}|\partial\phi(t,x)|_s \le \sum_{|\alpha|\le s} |\partial(\Gamma^\alpha\phi)(t,x)| \le C_s|\partial\phi(t,x)|_s \tag{2.2}$$

for any smooth function ϕ.

For the decay estimate of solutions to Klein-Gordon equations, we use the following estimate in Georgiev [4]:

Lemma 2.1. *Let χ_j with $j \ge 0$ be non-negative $C_0^\infty(\mathbb{R})$-functions satisfying $\sum_{j=0}^\infty \chi_j(\tau) = 1$ for $\tau \ge 0$, $\operatorname{supp}\chi_j \subset [2^{j-1}, 2^{j+1}]$ for $j \ge 1$, and $\operatorname{supp}\chi_0 \cap [0,\infty) \subset [0, 2]$. Let $m > 0$, and v be a smooth solution to*

$$\left(\Box + m^2\right) v(t,x) = \Phi(t,x),\ (t,x) \in (0,\infty)\times\mathbb{R}^3.$$

Then there exists a positive constant $C = C(m)$ such that we have

$$\begin{aligned} \langle t+|x|\rangle^{3/2}|v(t,x)| \le & C\sum_{j=0}^\infty \sum_{|\alpha|\le 4} \sup_{\tau\in[0,t]} \chi_j(\tau) \big\|\langle \tau + |\cdot|\rangle \Gamma^\alpha \Phi(\tau,\cdot)\big\|_{L^2(\mathbb{R}^3)} \\ & + C\sum_{j=0}^\infty \sum_{|\alpha|\le 5} \left\|\langle\cdot\rangle^{3/2}\chi_j(|\cdot|)\Gamma^\alpha v(0,\cdot)\right\|_{L^2(\mathbb{R}^3)} \end{aligned} \tag{2.3}$$

for $(t,x) \in (0,\infty)\times\mathbb{R}^3$, provided that the right-hand side of (2.3) *is finite.*

We use the weighted L^∞–L^∞ estimates for wave equations. We put

$$W_-(t,r) := \min\{\langle r\rangle, \langle t-r\rangle\}. \tag{2.4}$$

Lemma 2.2. *Let w be a smooth solution to*

$$\Box w(t,x) = 0,\ (t,x) \in (0,\infty)\times\mathbb{R}^3$$

with initial data $\big(w(0), (\partial_t w)(0)\big) = \big(w^{(0)}, w^{(1)}\big)$.

Let $\kappa > 0$. Then, there exists a positive constant $C = C(\kappa)$ such that

$$\begin{aligned}&\langle t+|x|\rangle \langle t-|x|\rangle^{\kappa} |w(t,x)| \\ &\qquad \le C \sup_{|y-x|\le t} \langle y\rangle^{1+\kappa} \left(\langle y\rangle \sum_{|\alpha|\le 1} \left|\left(\partial_x^{\alpha} w^{(0)}\right)(y)\right| + |y|\left|w^{(1)}(y)\right| \right)\end{aligned}$$

for $(t,x) \in (0,\infty)\times\mathbb{R}^3$. Here $\partial_x = (\partial_1, \partial_2, \partial_3)$, and we have used the standard notation of multi-indices.

For the proof, see Asakura [1] (see also [9] for the above expression).

The next decay estimates for inhomogeneous wave equations are due to Kubota-Yokoyama [13] (see also [7] for the expression below).

Lemma 2.3. *Let w be a smooth solution to*

$$\Box w(t,x) = \Psi(t,x),\ (t,x) \in (0,\infty)\times\mathbb{R}^3$$

with initial data $w = \partial_t w = 0$ at $t = 0$.

Suppose that $\rho \ge 0$ and $\kappa, \nu > 0$. Then there exists a positive constant $C = C(\rho,\kappa,\nu)$ such that

$$\begin{aligned}&\langle t+|x|\rangle^{1-\rho} \langle t-|x|\rangle^{\kappa} |w(t,x)| \\ &\quad \le C \sup_{\tau\in[0,t]} \sup_{|y-x|\le t-\tau} |y| \langle \tau+|y|\rangle^{1+\kappa+\nu-\rho} W_-(\tau,|y|)^{1-\nu} |\Psi(\tau,y)|, \\ &\langle t+|x|\rangle^{-\rho} \langle x\rangle \langle t-|x|\rangle^{\kappa+1} |\partial w(t,x)| \\ &\quad \le C \sup_{\tau\in[0,t]} \sup_{|y-x|\le t-\tau} |y| \langle \tau+|y|\rangle^{1+\kappa+\nu-\rho} W_-(\tau,|y|)^{1-\nu} \sum_{|\alpha|+|\beta|\le 1} |\partial^{\alpha}\Omega^{\beta}\Psi(\tau,y)|\end{aligned}$$

for $(t,x) \in (0,\infty)\times\mathbb{R}^3$.

The following Sobolev type inequality, proved in Klainerman [12], will be used to combine decay estimates with the energy estimates:

Lemma 2.4. *There is a positive constant C such that we have*

$$\sup_{x\in\mathbb{R}^3} \langle x\rangle |\varphi(x)| \le C \sum_{|\alpha|+|\beta|\le 2} \left\|\partial_x^{\alpha}\Omega^{\beta}\varphi\right\|_{L^2(\mathbb{R}^3)} \tag{2.5}$$

for any smooth function φ on $\mathbb{R}^3$, provided that the right-hand side of (2.5) is finite.

§ 3. The Null Conditions and Algebraic Normal Forms

For smooth functions ϕ and ψ, we define the *null forms*

$$Q_0(\phi,\psi) = (\partial_t\phi)(\partial_t\psi) - (\nabla_x\phi)\cdot(\nabla_x\psi), \tag{3.1}$$

$$Q_{ab}(\phi,\psi) = (\partial_a\phi)(\partial_b\psi) - (\partial_b\phi)(\partial_a\psi), \quad 0 \le a,b \le 3. \tag{3.2}$$

Q_{ab}'s are sometimes called the *strong null forms*. The null forms are closely related to the null condition for $F_{\mathrm{W}}^{\mathrm{W}}$. Indeed, (N1), the null condition for $F_{\mathrm{W}}^{\mathrm{W}}$, is satisfied if and only if

$$F_j^{kl}[w_k, w_l] = C_j^{kl} Q_0(w_k, w_l) + \sum_{0 \le a < b \le 3} C_j^{kl,ab} Q_{ab}(w_k, w_l) \tag{3.3}$$

for $N_0 + 1 \le j, k, l \le N$ with some constants C_j^{kl} and $C_j^{kl,ab}$, where $F_j^{kl}[w_k, w_l]$ is from (1.5)–(1.6).

Let us introduce

$$G_m(\phi, \psi) = Q_0(\phi, \psi) - m^2 \phi\psi,$$
$$H_a(\phi, \psi) = (\partial_a \phi)\psi + \phi(\partial_a \psi)$$

for $m \ge 0$ and $a = 0, 1, 2, 3$. Then we can show that the null condition for $F_{\mathrm{W}}^{\mathrm{K}}$ is satisfied if and only if

$$\begin{aligned} F_j^{kl}[v_k, v_l] = {} & C_j^{kl} G_{m_k}(v_k, v_l) + \sum_{a=0}^{3} C_j^{kl,a} H_a(v_k, v_l) \\ & + \sum_{0 \le a < b \le 3} C_j^{kl,ab} Q_{ab}(v_k, v_l) \end{aligned} \tag{3.4}$$

for $N_0 + 1 \le j \le N$ and $1 \le k, l \le N_0$ with $m_k = m_l$, where C_j^{kl}, $C_j^{kl,a}$, and $C_j^{kl,ab}$ are appropriate constants, and $F_j^{kl}[v_k, v_l]$ is from (1.5)–(1.6).

Remark. The null conditions for $F_{\mathrm{W}}^{\mathrm{W}}$ and $F_{\mathrm{W}}^{\mathrm{K}}$ are closely related to each other. Indeed, we can write (3.3) and (3.4) in a unified way:

$$\begin{aligned} F_j^{kl}[u_k, u_l] = {} & C_j^{kl} G_{m_k}(u_k, u_l) + \sum_{a=0}^{3} D_j^{kl,a} H_{m_k,a}(u_k, u_l) \\ & + \sum_{0 \le a < b \le 3} C_j^{kl,ab} Q_{ab}(u_k, u_l) \end{aligned} \tag{3.5}$$

for $N_0 + 1 \le j \le N$ and $1 \le k, l \le N$ with $m_k = m_l$, where

$$H_{m,a}(\phi, \psi) = (\partial_a \phi)(m\psi) + (m\phi)(\partial_a \psi).$$

When $N_0 + 1 \le k, l \le N$ and thus $m_k = m_l = 0$ in (3.5), we have $G_0(u_k, u_l) = Q_0(w_k, w_l)$, $H_{0,a}(u_k, u_l) = 0$, and $Q_{ab}(u_k, u_l) = Q_{ab}(w_k, w_l)$. Hence (3.5) coincides with (3.3). For $1 \le k, l \le N_0$ with $m_k = m_l (> 0)$, setting $C_j^{kl,a} = D_j^{kl,a} m_k$, we obtain (3.4) from (3.5).

For the strong null forms Q_{ab}, it is known that

$$|Q_{ab}(u_k,u_l)|_s \le C_s \langle t+r\rangle^{-1}\left(|u|_{[s/2]+1}|\partial u|_s + |\partial u|_{[s/2]}|u|_{s+1}\right) \tag{3.6}$$

for any non-negative integer s, where C_s is a positive constant depending only on s (see [5] for instance). In [5], we have obtained a similar, but slightly weaker estimate for Q_0. In this paper, we will not use such estimate for Q_0; instead, we use the method of algebraic normal forms to treat the null form Q_0, as well as G_m and H_a.

Here we briefly explain the method of algebraic normal forms (see Katayama-Ozawa-Sunagawa [8] for more details). For $1 \le j \le N$, we put

$$\Box_j = \Box + m_j^2.$$

We also set

$$(\eta_0,\eta_1,\eta_2,\eta_3) = (1,-1,-1,-1).$$

Since we have $(\partial_b\phi)(\partial_b\partial_a\psi) = (\partial_a\phi)(\partial_b^2\psi) + Q_{ba}(\phi,\partial_b\psi)$, we get

$$Q_0(\phi,\partial_a\psi) = (\partial_a\phi)(\Box\psi) + \sum_{b=0}^{3}\eta_b Q_{ba}(\phi,\partial_b\psi) \tag{3.7}$$

for $0 \le a \le 3$ and smooth functions ϕ,ψ.

Let $1 \le j,k,l \le N$ and $|\alpha|,|\beta| \le 1$. We put

$$U_{kl}^{\alpha\beta} := \Big((\partial^\alpha u_k)(\partial^\beta u_l) \;\; Q_0(\partial^\alpha u_k,\partial^\beta u_l)\Big).$$

It follows from simple calculation and (3.7) that

$$\Box_j U_{kl}^{\alpha\beta} = U_{kl}^{\alpha\beta}\mathcal{A}^{jkl} + \left(R_1^{kl\alpha\beta} \;\; R_2^{kl\alpha\beta}\right) \tag{3.8}$$

with

$$\mathcal{A}^{jkl} = \begin{pmatrix} m_j^2 - m_k^2 - m_l^2 & 2m_k^2 m_l^2 \\ 2 & m_j^2 - m_k^2 - m_l^2 \end{pmatrix},$$

$$R_1^{kl\alpha\beta} = (\partial^\alpha\Box_k u_k)(\partial^\beta u_l) + (\partial^\alpha u_k)(\partial^\beta \Box_l u_l), \tag{3.9}$$

$$\begin{aligned} R_2^{kl\alpha\beta} = {} & Q_0(\partial^\alpha\Box_k u_k,\partial^\beta u_l) + Q_0(\partial^\alpha u_k,\partial^\beta\Box_l u_l) + 2(\partial^\alpha\Box_k u_k)(\partial^\beta\Box_l u_l) \\ & - 2m_k^2(\partial^\alpha u_k)(\partial^\beta\Box_l u_l) - 2m_l^2(\partial^\alpha\Box_k u_k)(\partial^\beta u_l) \\ & + \sum_{a,b=0}^{3}\eta_a\eta_b Q_{ba}(\partial_a\partial^\alpha u_k,\partial_b\partial^\beta u_l). \end{aligned} \tag{3.10}$$

Using these formulas, we obtain the following key lemma:

Lemma 3.1. *Let u be the solution to* (1.1). *We assume that $|\alpha|, |\beta| \le 1$, and $1 \le j, k, l \le N$. If $\mathcal{A}^{jkl}$ is invertible, then there exist two constants c_{jkl} and d_{jkl} such that, writing*

$$(\partial^\alpha u_k)(\partial^\beta u_l) = \Box_j \left(c_{jkl}(\partial^\alpha u_k)(\partial^\beta u_l) + d_{jkl} Q_0(\partial^\alpha u_k, \partial^\beta u_l) \right) + \mathcal{R}^{\alpha\beta}_{jkl},$$

we have

$$\begin{aligned} |\mathcal{R}^{\alpha\beta}_{jkl}|_s \le {} & C_s \big(|u|_{[s/2]+2} |F|_{s+2} + |F|_{[s/2]+2} (|u|_{s+2} + |F|_{s+2}) \big) \\ & + C_s \langle t + r \rangle^{-1} |\partial u|_{[s/2]+2} |\partial u|_{s+2} \end{aligned}$$

for any non-negative integer s, where C_s is a positive constant.

Proof. By (3.8), we have

$$U^{\alpha\beta}_{kl} = \Box_j U^{\alpha\beta}_{kl} (\mathcal{A}^{jkl})^{-1} - \begin{pmatrix} R_1^{kl\alpha\beta} & R_2^{kl\alpha\beta} \end{pmatrix} (\mathcal{A}^{jkl})^{-1}.$$

Writing

$$(\mathcal{A}^{jkl})^{-1} = \begin{pmatrix} c_{jkl} & * \\ d_{jkl} & * \end{pmatrix},$$

we get

$$(\partial^\alpha u_k)(\partial^\beta u_l) = \Box_j \big(c_{jkl}(\partial^\alpha u_k)(\partial^\beta u_l) + d_{jkl} Q_0(\partial^\alpha u_k, \partial^\beta u_l) \big) + \mathcal{R}^{\alpha\beta}_{jkl}$$

with $\mathcal{R}^{\alpha\beta}_{jkl} = -c_{jkl} R_1^{kl\alpha\beta} - d_{jkl} R_2^{kl\alpha\beta}$. If we replace $\Box_k u_k$ and $\Box_l u_l$ in (3.9) and (3.10) with $F_k(u, \partial u)$ and $F_l(u, \partial u)$, respectively, we obtain the desired estimate for $|\mathcal{R}^{\alpha\beta}_{jkl}|_s$ with the help of (3.6). $\Box$

When we derive decay estimates for local solutions, Lemma 3.1 can be used to replace terms like $(\partial^\alpha u_k)(\partial^\beta u_l)$ in F_j with harmless terms, provided that $\det \mathcal{A}^{jkl} \ne 0$. For example, if $m_j > 0$ and $m_k = m_l = 0$, we have $\det \mathcal{A}^{jkl} = m_j^4 > 0$. Hence we can replace $F_{\mathrm{K}}^{\mathrm{W}}$ with harmless terms in the decay estimates below.

When $m_j = 0$, we have $\det \mathcal{A}^{jkl} = (m_k^2 - m_l^2)^2$. Hence, if $m_k \ne m_l$, we can replace $(\partial^\alpha u_k)(\partial^\beta u_l)$ in $F_{\mathrm{W}}^{\mathrm{q}}$ with harmless terms. When $m_j = m_k = m_l = 0$, we have $\det \mathcal{A}^{jkl} = 0$, but (N1), the null condition for $F_{\mathrm{W}}^{\mathrm{W}}$, is assumed for these terms, and the null form $Q_0(u_k, u_l)$ in $F_{\mathrm{W}}^{\mathrm{W}}$ can be similarly replaced with better terms.

Lemma 3.2. *Let u be the solution to* (1.1). *For $N_0 + 1 \le k, l \le N$, if we write*

$$Q_0(w_k, w_l) = \frac{1}{2} \Box (w_k w_l) + \mathcal{R}_{kl},$$

then, for any non-negative integer s we have

$$\big| \mathcal{R}_{kl} \big|_s \le C_s \big(|w|_{[s/2]} |F|_s + |F|_{[s/2]} |w|_s \big)$$

with a positive constant C_s depending only on s.

Proof. Since we have

$$\Box(w_k w_l) = (\Box w_k)w_l + w_k(\Box w_l) + 2Q_0(w_k, w_l),$$

we can easily show the desired result. $\Box$

When $m_j = 0$ and $m_k = m_l > 0$, we also have $\det \mathcal{A}^{jkl} = 0$. For these terms we have assumed (N2), the null condition for $F_{\mathrm{W}}^{\mathrm{K}}$, and we can replace G_{m_k} and H_a with harmless terms by the following lemma.

Lemma 3.3. *Let u be the solution to* (1.1). *For $1 \le k, l \le N_0$ with $m_k = m_l$, if we write*

$$G_{m_k}(v_k, v_l) = \frac{1}{2}\Box(v_k v_l) + \mathcal{R}_{kl},$$
$$H_a(v_k, v_l) = -\frac{1}{2m_k^2}\Box\bigl(v_k(\partial_a v_l)\bigr) + \mathcal{R}_{kl}^a,$$

then, for any non-negative integer s, we have

$$\bigl|\mathcal{R}_{kl}\bigr|_s + \sum_{a=0}^{3}\bigl|\mathcal{R}_{kl}^a\bigr|_s \le C_s\bigl(|v|_{[s/2]+1}|F|_{s+1} + |F|_{[s/2]+1}|v|_{s+1}\bigr) + C_s\,\langle t+r\rangle^{-1}\,|v|_{[s/2]+2}|v|_{s+2}$$

with a positive constant C_s depending only on s.

Proof. Using (3.7), we obtain

$$\Box\bigl(v_k v_l\bigr) = (\Box_k v_k)v_l + v_k(\Box_l v_l) + 2G_{m_k}(v_k, v_l),$$
$$\Box\bigl(v_k(\partial_a v_l)\bigr) = (\Box_k v_k)(\partial_a v_l) + v_k(\partial_a \Box_l v_l) + 2(\partial_a v_k)(\Box_l v_l) + 2\sum_{b=0}^{3}\eta_b Q_{ba}(v_k, \partial_b v_l) - 2m_k^2 H_a(v_k, v_l).$$

Now we can show the result in a similar fashion to the proof of Lemma 3.1. $\Box$

§ 4. Proof of Theorem 1.2

Recall that $|\cdot|_s$ and $\|\cdot\|_s$ are defined by (2.1).

For a smooth solution $u=(v,w)$ to (1.1)–(1.2) in $[0,T)\times\mathbb{R}^3$, we define

$$\begin{aligned}E[u](T):=&\sup_{t\in[0,T)}\langle t\rangle^{-\lambda}\left(\|v(t)\|_{2I}+\|\partial u(t)\|_{2I}\right)\\&+\sup_{(t,x)\in[0,T)\times\mathbb{R}^3}\langle t+|x|\rangle^{3/2}|v(t,x)|_{I+1}\\&+\sup_{(t,x)\in[0,T)\times\mathbb{R}^3}\langle t+|x|\rangle\langle t-|x|\rangle^{\kappa}|w(t,x)|_{I+1}\\&+\sup_{(t,x)\in[0,T)\times\mathbb{R}^3}\langle x\rangle\langle t-|x|\rangle^{\kappa+1}|\partial w(t,x)|_{I},\end{aligned}$$

where λ and κ are positive constants. I is a sufficiently large integer. Observe that there is a positive constant C such that

$$C^{-1}\langle t+|x|\rangle W_-(t,|x|)\le\langle x\rangle\langle t-|x|\rangle\le C\langle t+|x|\rangle W_-(t,|x|)$$

for $(t,x)\in[0,\infty)\times\mathbb{R}^3$, where W_- is given by (2.4).

Let $M>0$ be sufficiently large, and ε be sufficiently small compared to M. We are going to prove that $E[u](T)\le M\varepsilon$ implies $E[u](T)\le M\varepsilon/2$, provided that λ and κ are appropriately chosen. Then, by the so-called bootstrap argument, we can show the global existence of the solution u, which satisfies $E[u](\infty)\le C\varepsilon$ for $0<\varepsilon\le\varepsilon_0$ with some positive constants C and ε_0. From this estimate, we immediately obtain (1.9), (1.14), and (1.15).

Proof of Theorem 1.2. Let $I\ge 10$, $0<\kappa<1/2$, and λ be small enough to satisfy $0<\lambda<\min\{(1/2)-\kappa,2\kappa,1/6\}$. We assume that we have $E[u](T)\le M\varepsilon$ with some large M and small ε. We assume that ε is sufficiently small compared to M, so that we have $M\varepsilon\le 1$ and $M^2\varepsilon\le 1$. In what follows, C stands for positive constants which are independent of M, ε, and T.

Step 1: Energy estimates. By applying the standard energy inequality for wave and Klein-Gordon equations to $(\Box+m_j^2)\Gamma^\alpha u_j=\Gamma^\alpha F_j(u,\partial u)$ with $|\alpha|\le 2I$, we obtain

$$\|v(t)\|_{2I}+\|\partial u(t)\|_{2I}\le C\left(\varepsilon+\int_0^t\|F(u,\partial u)(\tau)\|_{2I}d\tau\right).\tag{4.1}$$

We are going to estimate $\|F(u,\partial u)(t)\|_{2I}$.

By Conditions (N1), (A1), and (A2), F^{q} is independent of w itself, and we obtain

$$\big\|\langle t+|\cdot|\rangle|F^{\mathrm{q}}(u,\partial u)(t)|_{2I}\big\|_{L^2}\le C\big\|\langle t+|\cdot|\rangle|(v,\partial u)|_I\big\|_{L^\infty}\|(v,\partial u)\|_{2I}\le CM^2\varepsilon^2\langle t\rangle^{\lambda}.$$

We have

$$\begin{aligned}\big\|\langle t+|\cdot|\rangle|F^{\mathrm{h}}(u,\partial u)(t)|_{2I}\big\|_{L^2}\le&\ C\big\|\langle t+|\cdot|\rangle|u|_{I+1}^2|(v,\partial u)|_{2I}\big\|_{L^2}\\&+C\big\|\langle t+|\cdot|\rangle|u|_{I+1}^2|w|_{2I}\big\|_{L^2}\\\le&\ CM^3\varepsilon^3\langle t\rangle^{\lambda-1}+CM\varepsilon\big\||u|_{I+1}|w|_{2I}\big\|_{L^2}.\end{aligned}$$

Since

$$|w|_{2I} \le C(|w| + |\Gamma w|_{2I-1}) \le C(|w| + \langle t+|x|\rangle\, |\partial w|_{2I-1}),$$

we get

$$\begin{aligned}\big\| |u|_{I+1}|w|_{2I}\big\|_{L^2} &\le C\big(\big\||u|_{I+1}|w|\big\|_{L^2} + \big\|\langle t+|\cdot|\rangle\, |u|_{I+1}\big\|_{L^\infty} \|\partial w\|_{2I-1}\big)\\ &\le C\left(M^2\varepsilon^2 \big\|\langle t+|\cdot|\rangle^{-2}\big\|_{L^2} + M^2\varepsilon^2 \langle t\rangle^\lambda\right)\\ &\le CM^2\varepsilon^2\langle t\rangle^{-1/2} + CM^2\varepsilon^2\langle t\rangle^\lambda \le CM^2\varepsilon^2\langle t\rangle^\lambda .\end{aligned}$$

Therefore we get

$$\big\|\langle t+|\cdot|\rangle\, |F(u,\partial u)(t)|_{2I}\big\|_{L^2} \le CM^2\varepsilon^2\langle t\rangle^\lambda , \tag{4.2}$$

which implies

$$\|F(u,\partial u)(t)\|_{2I} \le \big\|\langle t+|\cdot|\rangle^{-1}\big\|_{L^\infty}\big\|\langle t+|\cdot|\rangle\, |F(u,\partial u)(t)|_{2I}\big\|_{L^2} \le CM^2\varepsilon^2\langle t\rangle^{\lambda-1} .$$

Now (4.1) yields

$$\|v(t)\|_{2I} + \|\partial u(t)\|_{2I} \le C\big(\varepsilon + M^2\varepsilon^2\big)\langle t\rangle^\lambda \le C\varepsilon\langle t\rangle^\lambda , \tag{4.3}$$

since $M^2\varepsilon \le 1$.

Step 2: Rough decay estimates. By Lemma 2.1 and (4.2), we obtain

$$\langle t+|x|\rangle^{3/2}\, |v(t,x)|_{2I-4} \le C\varepsilon + CM^2\varepsilon^2\langle t\rangle^\lambda \le C\varepsilon\langle t\rangle^\lambda , \tag{4.4}$$

because we have

$$\sum_{j=0}^{\infty} \sup_{\tau\in[0,t]} \chi_j(\tau)\langle\tau\rangle^\lambda \le C\langle t\rangle^\lambda .$$

For $\rho \ge 0$, $\mu,\nu > 0$, a non-negative integer s, and a smooth function $\phi = \phi(t,x)$, we put

$$\begin{aligned}\mathcal{A}_{\rho,\mu,s}[\phi] &:= \sup_{(t,x)\in[0,T)\times\mathbb{R}^3} \langle t+|x|\rangle^{1-\rho}\langle t-|x|\rangle^{\mu}\, |\phi(t,x)|_s,\\ \mathcal{B}_{\rho,\mu,\nu,s}[\phi] &:= \sup_{(t,x)\in[0,T)\times\mathbb{R}^3} |x|\,\langle t+|x|\rangle^{1+\mu+\nu-\rho}\, W_-(t,|x|)^{1-\nu}|\phi(t,x)|_s.\end{aligned}$$

Then, for a positive integer s, Lemmas 2.2 and 2.3 yield

$$\mathcal{A}_{\rho,\mu,s}[\phi] \le C\big(\varepsilon + \mathcal{B}_{\rho,\mu,\nu,s}[\Box\phi]\big) \tag{4.5}$$

for $\rho \ge 0$ and $\mu,\nu > 0$, provided that $\phi(0)$ and $(\partial_t\phi)(0)$ are rapidly decreasing, and their amplitude is of order ε. We also have

$$\begin{aligned}\sup_{(t,x)\in[0,T)\times\mathbb{R}^3} &\langle t+|x|\rangle^{-\rho}\langle x\rangle\langle t-|x|\rangle^{\mu+1}\, |\partial\phi(t,x)|_{s-1}\\ &\le C\big(\varepsilon + \mathcal{B}_{\rho,\mu,\nu,s}[\Box\phi]\big).\end{aligned} \tag{4.6}$$

Let $0 < \delta < (1/2) - \kappa - \lambda$. For $s \le 2I$, we have

$$
\begin{aligned}
(4.7)\qquad |F^{\mathrm{h}}|_s &\le C|u|_{I+1}^2\big(|w|_s + |(v,\partial u)|_s\big)\\
&\le CM^2\varepsilon^2 \langle t+|x|\rangle^{-2} W_-(t,|x|)^{-2\kappa}\big(|w|_s + |(v,\partial u)|_s\big)\\
&\le CM^2\varepsilon^2 \langle t+|x|\rangle^{-2} W_-(t,|x|)^{-2\kappa}|(v,\partial u)|_s\\
&\quad + CM^2\varepsilon^2 \langle t+|x|\rangle^{\rho-3} W_-(t,|x|)^{-2\kappa-\delta}\mathcal{A}_{\rho,\delta,s}[w].
\end{aligned}
$$

By (N1), (A1), and (A2), F^{q} is independent of w itself, and we get

$$
\begin{aligned}
(4.8)\qquad |F^{\mathrm{q}}|_s &\le C|(v,\partial u)|_{[s/2]}|(v,\partial u)|_s\\
&\le CM\varepsilon \langle t+|x|\rangle^{-1} W_-(t,|x|)^{-1/2}|(v,\partial u)|_s.
\end{aligned}
$$

By Lemma 2.4 and (4.3), we obtain

$$
|(v,\partial u)(t,x)|_{2I-2} \le C\langle x\rangle^{-1}\|(v,\partial u)(t)\|_{2I} \le C\varepsilon\langle x\rangle^{-1}\langle t\rangle^{\lambda}.
$$

Therefore we get

$$
\begin{aligned}
|F_{\mathrm{W}}|_{2I-2} &\le CM\varepsilon^2\langle x\rangle^{-1}\langle t+|x|\rangle^{\lambda-1} W_-(t,|x|)^{-1/2}\\
&\quad + CM^2\varepsilon^2\langle t+|x|\rangle^{\lambda+\delta-(5/2)} W_-(t,|x|)^{-2\kappa-\delta}\mathcal{A}_{\lambda+\delta+(1/2),\delta,2I-2}[w],
\end{aligned}
$$

which leads to

$$
\mathcal{B}_{\lambda+\delta+(1/2),\delta,1/2,2I-2}[F_{\mathrm{W}}] \le CM\varepsilon^2 + CM^2\varepsilon^2\mathcal{A}_{\lambda+\delta+(1/2),\delta,2I-2}[w],
$$

since we have $0<\delta<1/2$. It follows from (4.5) that

$$
\mathcal{A}_{\lambda+\delta+(1/2),\delta,2I-2}[w] \le C\left(\varepsilon + M^2\varepsilon^2\mathcal{A}_{\lambda+\delta+(1/2),\delta,2I-2}[w]\right),
$$

and if we choose sufficiently small ε satisfying $CM^2\varepsilon^2 \le 1/2$, we get

$$
\begin{aligned}
(4.9)\qquad &\sup_{(t,x)\in[0,T)\times\mathbb{R}^3}\langle t+|x|\rangle^{(1/2)-\lambda-\delta}\langle t-|x|\rangle^{\delta}|w(t,x)|_{2I-2}\\
&= \mathcal{A}_{\lambda+\delta+(1/2),\delta,2I-2}[w] \le C\varepsilon.
\end{aligned}
$$

By (4.6), we also have

$$
\begin{aligned}
(4.10)\qquad |\partial w(t,x)|_{2I-3} &\le C\varepsilon\langle t+|x|\rangle^{\lambda+\delta+(1/2)}\langle x\rangle^{-1}\langle t-|x|\rangle^{-1-\delta}\\
&\le C\varepsilon\langle t+|x|\rangle^{\lambda+\delta-(1/2)} W_-(t,|x|)^{-1-\delta}.
\end{aligned}
$$

Step 3: Better decay estimates through the method of algebraic normal forms. We use the algebraic normal forms to replace $F_{\mathrm{W}}^{\mathrm{q}}$ with harmless terms: Terms $(\partial^\alpha v_j)(\partial^\beta v_k)$ with $m_j \ne m_k$ in $F_{\mathrm{W}}^{\mathrm{K}}$ and all the terms in $F_{\mathrm{W}}^{\mathrm{KW}}$ can be treated by

Lemma 3.1, while the other terms in $F_{\mathrm{W}}^{\mathrm{K}}$ have the expression (3.4) by the assumption (N2), and the terms on the right-hand side of (3.4), except the strong null forms, can be treated by Lemma 3.3. We can use Lemma 3.2 to replace $Q_0(w_j, w_k)$ in $F_{\mathrm{W}}^{\mathrm{W}}$, and only the strong null forms are left in $F_{\mathrm{W}}^{\mathrm{W}}$. Because of the enhanced decay estimate (3.6), we can consider the strong null forms as harmless terms. In this way, we can find $P_{\mathrm{W}} = \big(P_j(u, \partial u, \partial^2 u)\big)_{N_0+1\le j\le N}$, whose components are homogeneous polynomials of degree 2 in $(u, \partial u, \partial^2 u)$, such that

$$\square(w - P_{\mathrm{W}}) = F_{\mathrm{W}}^{\mathrm{h}}(u, \partial u) + \mathcal{R}_{\mathrm{W}},$$

where, for $s \le 2I - 4$, we have

$$\begin{aligned}|\mathcal{R}_{\mathrm{W}}|_s \le{}& C\,\langle t+|x|\rangle^{-1}\big(|w|_{I+1}|\partial w|_s + |\partial w|_I|w|_{s+1} + |(v,\partial u)|_I|(v,\partial u)|_{s+2}\big)\\ &+ C\,\big(|u|_{I+1}|F|_{s+2} + |F|_I\big(|u|_{s+2} + |F|_{s+2}\big)\big).\end{aligned}$$

From (4.4) and (4.10), we get

$$|(v,\partial u)|_{2I-5} \le C\varepsilon\,\langle t+|x|\rangle^{\lambda+\delta-(1/2)}\,W_-(t,|x|)^{-1-\delta}.$$

Hence, using also (4.9), we obtain

$$\begin{aligned}|w|_{I+1}|\partial w|_{2I-7} + |\partial w|_I|w|_{2I-6} &\le CM\varepsilon^2\,\langle t+|x|\rangle^{\lambda+\delta-(3/2)}\,W_-(t,|x|)^{-1-\kappa-\delta},\\ |(v,\partial u)|_I|(v,\partial u)|_{2I-5} &\le CM\varepsilon^2\,\langle t+|x|\rangle^{\lambda+\delta-(3/2)}\,W_-(t,|x|)^{-(3/2)-\delta}.\end{aligned}$$

With the help of (4.7) and (4.8), we also have

$$\begin{aligned}|F|_{2I-5} \le{}& CM\varepsilon^2\,\langle t+|x|\rangle^{\lambda+\delta-(3/2)}\,W_-(t,|x|)^{-(3/2)-\delta}\\ &+ CM^2\varepsilon^3\,\langle t+|x|\rangle^{\lambda+\delta-(5/2)}\,W_-(t,|x|)^{-1-2\kappa-\delta}\\ &+ CM^2\varepsilon^3\,\langle t+|x|\rangle^{\lambda+\delta-(5/2)}\,W_-(t,|x|)^{-2\kappa-\delta}\\ \le{}& CM\varepsilon^2\,\langle t+|x|\rangle^{\lambda+\delta-(3/2)}\,W_-(t,|x|)^{-1-\kappa-\delta},\\ |F|_I \le{}& CM^2\varepsilon^2\,\langle t+|x|\rangle^{-2}\,W_-(t,|x|)^{-1} + CM^3\varepsilon^3\,\langle t+|x|\rangle^{-3}\,W_-(t,|x|)^{-3\kappa}\\ \le{}& CM^2\varepsilon^2\,\langle t+|x|\rangle^{-2}\,W_-(t,|x|)^{-1}.\end{aligned}$$

To sum up, we obtain

$$|\mathcal{R}_{\mathrm{W}}|_{2I-7} \le CM\varepsilon^2\,\langle t+|x|\rangle^{\lambda+\delta-(5/2)}\,W_-(t,|x|)^{-1-\delta}.$$

On the other hand, (4.7) yields

$$\begin{aligned}|F_{\mathrm{W}}^{\mathrm{h}}|_{2I-7} \le{}& CM^2\varepsilon^3\,\langle t+|x|\rangle^{\lambda+\delta-(5/2)}\,W_-(t,|x|)^{-1-2\kappa-\delta}\\ &+ CM^2\varepsilon^2\,\langle t+|x|\rangle^{-3}\,W_-(t,|x|)^{-3\kappa}\mathcal{A}_{0,\kappa,2I-7}[w].\end{aligned}$$

Therefore, if we set $\nu=(1/2)-\kappa-\lambda-\delta$, we get

$$\mathcal{B}_{0,\kappa,\nu,2I-7}[\Box(w-P_{\mathrm{W}})]\le C\varepsilon+CM^2\varepsilon^2\mathcal{A}_{0,\kappa,2I-7}[w].$$

As

$$\mathcal{A}_{0,\kappa,2I-7}[P_{\mathrm{W}}]\le\sup_{(t,x)}\bigl(\langle t+|x|\rangle\langle t-|x|\rangle^{\kappa}|u(t,x)|_I\bigr)|u(t,x)|_{2I-5}\le CM\varepsilon^2,$$

(4.5) implies

$$\mathcal{A}_{0,\kappa,2I-7}[w]\le\mathcal{A}_{0,\kappa,2I-7}[P_{\mathrm{W}}]+\mathcal{A}_{0,\kappa,2I-7}[w-P_{\mathrm{W}}]\le C\varepsilon+CM^2\varepsilon^2\mathcal{A}_{0,\kappa,2I-7}[w].$$

If ε is sufficiently small to satisfy $CM^2\varepsilon^2\le 1/2$, we obtain

$$\sup_{(t,x)\in[0,T)\times\mathbb{R}^3}\langle t+|x|\rangle\langle t-|x|\rangle^{\kappa}|w(t,x)|_{2I-7}=\mathcal{A}_{0,\kappa,2I-7}[w]\le C\varepsilon. \tag{4.11}$$

Since it follows from (4.4), (4.10), and (4.11) that

$$\begin{aligned}|\partial P_{\mathrm{W}}|_{2I-8}&\le C(|u|_{I+1}|\partial u|_{2I-6}+|\partial u|_I\bigl(|w|_{2I-8}+|(v,\partial u)|_{2I-7}\bigr)\\&\le CM\varepsilon^2\langle t+|x|\rangle^{\lambda+\delta-(3/2)}W_-(t,|x|)^{-1-\kappa-\delta}\\&\le CM\varepsilon^2\langle t+|x|\rangle^{-1-\kappa}W_-(t,|x|)^{-1-\kappa}\\&\le CM\varepsilon^2\langle x\rangle^{-1-\kappa}\langle t-|x|\rangle^{-1-\kappa},\end{aligned}$$

(4.6) yields

$$\sup_{(t,x)\in[0,T)\times\mathbb{R}^3}\langle x\rangle\langle t-|x|\rangle^{1+\kappa}|\partial w(t,x)|_{2I-8}\le C\varepsilon. \tag{4.12}$$

We use the algebraic normal forms to replace $F_{\mathrm{K}}^{\mathrm{W}}$ with harmless terms: Using Lemma 3.1, we can find $P_{\mathrm{K}}=\bigl(P_j(w,\partial w,\partial^2 w)\bigr)_{1\le j\le N_0}$, whose components are homogeneous polynomials of degree 2 in $(w,\partial w,\partial^2 w)$, such that

$$(\Box+m_j^2)(v_j-P_j)=F_j^{\mathrm{K}}(v,\partial v)+F_j^{\mathrm{KW}}(u,\partial u)+F_j^{\mathrm{h}}(u,\partial u)+\mathcal{R}_j$$

for $1\le j\le N_0$, where

$$\bigl|\mathcal{R}_j\bigr|_{2I-5}\le C\bigl(|u|_{I+1}^2|(u,\partial u)|_{2I-3}+\langle t+r\rangle^{-1}|\partial u|_I|\partial u|_{2I-3}\bigr).$$

It follows from (4.9) that

$$\begin{aligned}\bigl\|\langle t+|\cdot|\rangle|u|_{I+1}^2|w|_{2I-3}\bigr\|_{L^2}&\le CM^2\varepsilon^3\|\langle t+|\cdot|\rangle^{\lambda+\delta-(3/2)}W_-(t,|\cdot|)^{-2\kappa-\delta}\|_{L^2}\\&\le CM^2\varepsilon^3\left(\int_0^\infty\langle t+r\rangle^{2\lambda+2\delta-1}W_-(t,r)^{-4\kappa-2\delta}dr\right)^{1/2}\\&\le CM^2\varepsilon^3\langle t\rangle^{\lambda-2\kappa}.\end{aligned}$$

Other terms in the upper bound for $\mathcal{R}_j$ can be easily treated, and we get

$$\left\| \langle t + |\cdot| \rangle \, |\mathcal{R}_j(t)|_{2I-5} \right\|_{L^2} \le CM\varepsilon^2 \langle t \rangle^{\lambda - 2\kappa}.$$

By (4.3), we have

$$\left\| \langle t + |\cdot| \rangle \, |F_{\mathrm{K}}^{\mathrm{K}}(v, \partial v)(t)|_{2I-5} \right\|_{L^2} \le \left\| \langle t + |\cdot| \rangle \, |v|_{I+1} \right\|_{L^\infty} \|v\|_{2I-4} \le CM\varepsilon^2 \langle t \rangle^{\lambda - (1/2)}.$$

We estimate $F_{\mathrm{K}}^{\mathrm{KW}}$ as

$$\begin{aligned} \left\| \langle t + |\cdot| \rangle \, |F_{\mathrm{K}}^{\mathrm{KW}}(u, \partial u)(t)|_{2I-5} \right\|_{L^2} &\le C \left\| \langle t + |\cdot| \rangle \, |v|_{2I-4} \right\|_{L^\infty} \|\partial w\|_{2I-5} \\ &\le C\varepsilon^2 \langle t \rangle^{2\lambda - (1/2)} \end{aligned}$$

by (4.3) and (4.4). Since F_j^{h} enjoys the same estimate as $\mathcal{R}_j$, we obtain

$$\left\| \langle t + |\cdot| \rangle \, |(\Box + m_j^2)(v_j - P_j)(t)|_{2I-5} \right\|_{L^2} \le C\varepsilon \langle t \rangle^{\min\{\lambda - 2\kappa, 2\lambda - (1/2)\}}, \tag{4.13}$$

and Lemma 2.1 yields

$$\langle t + |x| \rangle^{3/2} |(v_j - P_j)(t, x)|_{2I-9} \le C\varepsilon,$$

because $\min\{\lambda - 2\kappa, 2\lambda - (1/2)\} < 0$. By (4.11), we have

$$\langle t + |x| \rangle^{3/2} |P_j|_{2I-9} \le C \langle t + |x| \rangle^{3/2} |w|_{2I-7}^2 \le C\varepsilon^2 \langle t + |x| \rangle^{-1/2},$$

and we obtain

$$\langle t + |x| \rangle^{3/2} |v(t, x)|_{2I-9} \le C\varepsilon. \tag{4.14}$$

Step 4: Conclusion. Since $I + 1 \le 2I - 9$ for $I \ge 10$, by (4.3), (4.11), (4.12), and (4.14), we have the following: For any $M > 0$, there is a positive constant C_0, which is independent of (M, ε, T), and a positive constant $\varepsilon_0(M)$, which depends on M, but is independent of T, such that we have

$$E[u](T) \le C_0 \varepsilon$$

for $0 < \varepsilon \le \varepsilon_0(M)$. If we choose M to satisfy $M \ge 2C_0$, then we have proved that $E[u](T) \le M\varepsilon$ implies $E[u](T) \le M/2$ for $0 < \varepsilon \le \varepsilon_0(M)$.

Now, as was mentioned above, the bootstrap argument implies the global existence of the solution u for small ε, and we see that the solution u satisfies $E[u](\infty) \le M\varepsilon$. This completes the proof. □

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